

Charge without charge, regular spherically symmetric solutions and the Einstein-Born-Infeld theory

D.J. Cirilo Lombardo

Bogoliubov Laboratory of Theoretical Physics
Joint Institute of Nuclear Research,
Dubna, Moscow Region, 141980, Russia*

June 19, 2009

Abstract

The aim of this paper is to continue the research of JMP 46, 042501 (2005) of regular static spherically symmetric spacetimes in Einstein-Born-Infeld theories from the point of view of the spacetime geometry and the electromagnetic structure. The energy conditions, geodesic completeness and the main features of the horizons of this spacetime are explicitly shown. A new static spherically symmetric dyonic solution in Einstein-Born-Infeld theory with similar good properties as in the regular pure electric and magnetic cases of our previous work, is presented and analyzed. Also, the circumvention of a version of "no go" theorem claiming the non existence of regular electric black holes and other electromagnetic static spherically configurations with regular center is explained by dealing with a more general statement of the problem.

1 The regularity problem

*e-mails:diego@thsun1.jinr.ru diego77jcl@yahoo.com.

Last years considerable interest in regular black hole solutions has been renewed⁶, and in particular, to those based in nonlinear electrodynamics (NED)^{7,8}. In our previous work¹ we presented a new *exact* spherically symmetric solution of the Einstein-Born-Infeld (EBI) equations. The metric is *regular* everywhere in the sense that was given by B. Hoffmann and L. Infeld⁴ in 1937 and, when the intrinsic mass of the system is zero, the EBI theory leads to identification of the gravitational with the electromagnetic mass. Explicitly the line element that we obtained in ref.¹ is

$$ds^2 = -e^{2\Lambda} dt^2 + e^{2\mathcal{F}(r)} \left[e^{-2\Lambda} (1 + r \partial_r \mathcal{F}(r))^2 dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right], \quad (1)$$

in particular the g_{tt} coefficient, takes the following form :

$$e^{2\Lambda} = 1 - \frac{2M}{Y} - \frac{2b^2 r_o^4}{3 \left(\sqrt{Y^4 + r_o^4} + Y^2 \right)} - \frac{4}{3} b^2 r_o^2 {}_2F_1 \left[1/4, 1/2, 5/4; - \left(\frac{Y}{r_o} \right)^4 \right] \quad (2)$$

here M is an integration constant, which can be interpreted as an intrinsic mass, and ${}_2F_1$ is the Gauss hypergeometric function¹⁶. Specifically, for the form of the $\mathcal{F}(r)$ defined by the expression (1), the coordinate function $Y(r)$ is related with the function $\mathcal{F}(r)$ as

$$Y(r)^2 \equiv \left[1 - \left(\frac{r_o}{a|r|} \right)^n \right]^{2m} r^2 = e^{2\mathcal{F}(r)} r^2 \quad (2.1)$$

Regarding the historical origin of the question, in 1937 B. Hoffmann and L. Infeld⁴ introduce a regularity condition on the new field theory of M. Born¹⁰ with the main idea of to solve the lack of uniqueness of the function action¹. They have already seen that the condition of regularity of the field

¹The new field theory initiated in 1934 by M. Born¹⁰ introduces in the classical equations of the electromagnetic field a characteristic length r_0 representing the radius of the elementary particle through the relation

$$r_0 = \sqrt{\frac{e}{b}}$$

where e is the elementary charge and b the fundamental field strength entering in a non-linear Lagrangian function. It was originally thought that the Lagrangian $\mathcal{L}_{BI} = \sqrt{-g} L_{BI} = \frac{b^2}{4\pi} \left\{ \sqrt{-g} - \sqrt{|\det(g_{\mu\nu} + b^{-1} F_{\mu\nu})|} \right\}$ was the simplest choice which would lead to a finite energy for an electric particle. This is, however, not the case. It is possible to find an infinite number of quite different action functions, each giving simple algebraic relations between the fields and each leading to a finite energy for an electric particle.

gives the restriction in the spherically symmetric electrostatic case $E_r = 0$ for $r = 0$.

In the general theory they applied the regularity condition not only to the $F_{\mu\nu}$ field but also to the $g_{\mu\nu}$ field. The regularity condition for the general theory was that:

Only those solutions of the field equations may have physical meaning for which space-time is everywhere regular and for which the $F_{\mu\nu}$ and the $g_{\mu\nu}$ fields and those of their derivatives which enter in the field equations and the conservation laws exist everywhere.

In the general relativity form of the original new field theory the requirement that there be no infinities in the $g_{\mu\nu}$ forces the identification of gravitational with electromagnetic mass. In⁴ B. Hoffmann and L. Infeld have used for such identification the line element of the well known monopole solution studied by B. Hoffmann³ in 1935

$$A(r) \equiv 1 - \frac{8\pi G}{r} \int_0^r \left[(r'^4 + 1)^{1/2} - r'^2 \right] dr'$$

that is originated by an EBI action as in equation (1) of ¹ (see footnote 1). This line element approximates the Schwarzschild form for r greater than the electronic radius but avoid the infinities of that line element for $r = 0$. However is still a singularity of conical type at the origin. When $r \rightarrow 0$ the above expression for A , gives

$$A \rightarrow (1 - 8\pi G) \equiv \beta$$

so ds^2 becomes

$$ds^2 = -\beta dt^2 + \beta^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)$$

Therefore the origin (it is, at $r = 0$) is a conical point and not regular. Note that, because the conical point, no coordinate can be introduced which will be non singular at $r = 0$ and derivatives are actually undefined at this point.

This problem with the conical singularities at $r = 0$, that destroy the regularity condition, makes that in the reference⁴ B. Hoffmann and L. Infeld change the action of the Born-Infeld form for other non-linear Lagrangian of logarithmic type, were the metric takes the form :

$$A(r) \equiv 1 - \frac{8\pi G}{r} \int_0^r r'^2 \text{Log} \left[\frac{r'^2}{r'^4 + 1} \right] dr'$$

The new logarithmic action does not presented such difficulties at $r = 0$, but only for the electric field the solution is allowed: there are not possibility of magnetic monopoles. Also, the $\mathbb{F}^{\mu\nu} \equiv \frac{\partial L}{\partial F_{\mu\nu}}$ field has a divergent behaviour at $r \rightarrow 0$. Therefore, the only requirement of regularity in order to ruled out a specific form of the Lagrangian seems to be *insufficient*. Moreover: in ref. ¹ we was able to solve the problem of regularity with the original (first principles) BI Lagrangian, that permits a symmetric treatment of magnetic and electric field configurations with a full generalized duality (as was also shown by the authors in¹²).

Returning to the summary of the main features of the line element (1), a regular, asymptotically flat solution with the electric field and energy-momentum tensor both regular, in the sense of B. Hoffmann and L. Infeld is when the exponent numbers of $Y(r)$ take the following particular values:

$$n = 3 \quad \text{and} \quad m = 1$$

and the integration constant M can have any value, not necessarily zero (this fact what not advertised by the authors in ¹. We will return to this point in Section 8). The spacetime is singularity free and geodesically complete. In this case and for $r \gg \frac{r_0}{a}$, we have the following asymptotic behaviour for $Y(r)$ and $-g_{tt}$, that does not depend on the a parameter

$$Y(r) \rightarrow r \quad \left(r \gg \frac{r_0}{a} \right)$$

$$e^{2\Lambda} \simeq 1 - \frac{2M}{r} - \frac{8b^2 r_o^4 K(1/2)}{3r_o r} + \frac{2b^2 r_o^4}{r^2} + \dots$$

As we pointed out in ref.¹, a distant observer will associate with this solution a total mass

$$M_{eff} = M + \frac{4b^2 r_o^4 K(1/2)}{3r_o}$$

total charge

$$Q^2 = 2b^2 r_o^2$$

being the electromagnetic mass

$$M_{el} = \frac{4b^2 r_o^4 K(1/2)}{3r_o} \tag{2.2}$$

Notice that the M_{el} is necessarily positive, which was not the case in the Schwarzschild line element where M is an integration constant. The important reason for to take the constant $M = 0$ is that we must regard the quantity (let us to restore by one moment the gravitational constant G)

$$4\pi G \int_{Y(r=0)}^{Y(r)} T_0^0(Y) Y^2 dY$$

as the *gravitational mass* causing the field at coordinate distance r from the pole. In our case T_0^0 is given by $-\frac{b^2}{4\pi} \left(1 - \sqrt{\left(\frac{r_0}{e\mathcal{G}}\right)^4 + 1}\right)$. This quantity is precisely M_{el} (in gravitational units) given by (2.2), the *total electromagnetic mass* within the sphere having its center at $r = 0$ and coordinate r . We will take $M = 0$ in the rest of the analysis being consistent with the gravitational-electromagnetic identification.

On the other hand, the function $Y(r)$ for the values of the m and n parameters given above has the following behaviour near of the origin

$$\text{for } a < 0 \quad \text{when } r \rightarrow 0, Y(r) \rightarrow \infty$$

$$\text{for } a > 0 \quad \text{when } r \rightarrow 0, Y(r) \rightarrow -\infty$$

Notice that the case $a > 0$ will be excluded because in any value $r_0 \rightarrow Y(r_0) = 0$, the electric field takes the limit value b and the condition $F_{01}|_{r=r_0} < b$ is violated (and the T_{00} diverges). For $M = 0$ and $a < 0$, expanding the hypergeometric function, we can see that the $-g_{tt}$ coefficient has the following behaviour near the origin

$$e^{2\Lambda} \simeq 1 - \frac{8b^2 r_o^4 K(1/2)}{3r_o} r^2 \left(\frac{|a|}{r_0}\right)^3 + 2b^2 r_o^4 r^4 \left(\frac{|a|}{r_0}\right)^6 + \dots$$

The metric and the energy-momentum tensor remain *both* regulars at the origin (it is: $g_{tt} \rightarrow -1, T_{\mu\nu} \rightarrow 0$ for $r \rightarrow 0$). It is not very difficult to check that (for $m = 1$ and $n = 3$) the maximum of the electric field (see figures) is not in $r = 0$, but in the *physical border* of the spherical configuration source of the electromagnetic fields (this point is located around $r_B = 2^{1/3} \frac{r_0}{|a|}$). It means that $Y(r)$ maps correctly the internal structure of the source in the similarly form that the quasiglobal coordinate of the reference¹⁷ for the global monopole in general relativity but, in contrast to the case pointed out in ref.¹⁷, $Y(r)$ is *function* from the strict mathematical point of view. The lack

of the conical singularities at the origin is because the very well description of the manifold in the neighborhood of $r = 0$ given by the function $Y(r)$.

Because g_{tt} is regular ($g_{tt} = -1$, at $r = 0$ and at $r = \infty$), its derivative must change sign. In the usual gravitational theory of general relativity the derivative of g_{tt} is proportional to the gravitational force which would act on a test particle in the Newtonian approximation. In Einstein-Born-Infeld theory with this new static solution, it is interesting to note that although this force is attractive for distances of the order $r_0 \ll r$, it is actually a repulsion for very small r . For r greater than r_0 , the line element closely approximates to the Schwarzschild form. Thus the regularity condition shows that the electromagnetic and gravitational mass are the same and, as in the Newtonian theory, we now have the result that the attraction is zero in the center of the spherical configuration source of the electromagnetic field.

The aim of this paper is to make complete the research of regular static spherically symmetric spacetimes in EBI theories of ref.¹ emphasizing the following points:

- the analysis of the monopole solution presented in the previous work from the physical and topological point of view,

- how a version of no go theorem^{7,8} claiming the non existence of regular electric charged black holes and other electric configurations with regular center is circumvented by dealing with a correct statement of the problem. In fact was clearly demonstrated in¹ and results above, that the Born-Infeld theory coupled with general relativity (EBI) lead regular solutions where the main assumptions to have account were:

- i) most general spherically symmetric line element as starting point, compatible with the symmetry of the an anisotropic fluid¹⁵ (the Born-Infeld energy-momentum tensor has this form)

- ii) strict physical requirements of regularity for the electric field

$$F_{01}|_{r=r_0} < b$$

$$F_{01}|_{r=0} = 0$$

$$F_{01}|_{r \rightarrow \infty} = 0 \quad \text{asymptotically Coulomb}$$

that fix the form and behaviour of function $\mathcal{F}(r)$ being as (2.1) where a is an arbitrary constant, and the exponents n and m will obey the following relation

$$mn > 1 \quad (m, n \in \mathbb{N})$$

with

$$0 < a < 1 \text{ or } -1 < a < 0$$

depending on $m(n)$ is even or odd and

$$a \neq 0$$

that put in sure and *guarantees* a consistent regularization condition not only for the electric (magnetic) field but for the energy-momentum tensor and the line element also.

- to show that the electric solution of ¹ satisfy the weak, dominant and strong energy conditions (WEC, DEC, SEC) and the Lagrangian has correct Maxwellian limit at $r \rightarrow 0$ without branches.

- a new static spherically symmetric dyonic solution in EBI theory with similar good properties as for the regular pure electric and magnetic cases, is presented and analyzed.

The organization of the paper is as follows: Section 2 is devoted to show that the solution of¹ satisfy all the energy conditions. In Section 3 the horizons structure of the singularity free and geodesically complete spacetime of ¹ is fully analyzed and some physical aspects concerning to the Born-Infeld radius² r_o , the absolute field² b and the cosmological constant¹¹ Λ are briefly discussed. In Section 4 we explain with some detail how a version of no go theorem claiming the non existence of regular electric charged black holes and other electric configurations with regular center is circumvented.

Section 5, 6 and 7 are devoted to the new regular spherically symmetric dyonic solution in EBI: statement of the problem, determination of the electromagnetic structure of this spherically symmetric configuration: field equations vs. energy-momentum conservation and aspects of the electromagnetic field of the dyon in comparison with the pure electric and magnetic regular cases of our previous reference ¹. Finally, in Section 8 a resume of the main results and a discussion about the regular spherically symmetric solutions in NED are given in the light of the new results and analysis presented here. The conventions are the same that in ¹, unless that we indicate the contrary.

2 Energy conditions

Is clearly important, in order to begin the study of any spacetime solution, consider firstly if violation of the called¹³ energy conditions certainly exists.

The first step to begin the analysis of the monopole solution of ¹ is to remember that the (adimensionalized) $\bar{Y}^2(r)$ function

$$\bar{Y}(r)^2 \equiv \left(\frac{Y}{r_o}\right)^2 = \left[1 - \left(\frac{r_o}{a|r|}\right)^n\right]^{2m} \left(\frac{r}{r_o}\right)^2 \quad (3)$$

with $a = -0,9$ and $m = 1$ and $n = 3$, never is zero with its range being

$$\bar{Y}^2\Big|_{min} \leq \bar{Y}^2 < \infty \quad (4)$$

$$\bar{Y}^2\Big|_{min} = 2.09987 \text{ corresponding to } \bar{r}_{min} = 1.39991$$

that is the point where the electric field and the energy-momentum tensor have the maximum value and obviously, the metric present its maximum curvature (from here, the dependence of Y on r is understood) . All the quantities of our solution presented in ¹ can be written in a more compact form through this function. Now, we put all the components of the energy momentum tensor in the orthonormal frame (that coincide with the energy density ρ and the pressures p_k) as a functions of $\bar{Y}^2(r)$, in order to easily analyze if the solution fulfill the energy conditions:

$$\rho = T_{00} = \frac{b^2}{4\pi} \left(\sqrt{(\bar{Y})^{-4} + 1} - 1 \right) \quad (5a)$$

$$p_{rad} = -\rho = T_{11} = \frac{b^2}{4\pi} \left(1 - \sqrt{(\bar{Y})^{-4} + 1} \right) \quad (5b)$$

$$p_{\perp} = T_{22} = T_{33} = \frac{b^2}{4\pi} \left(1 - \frac{1}{\sqrt{(\bar{Y})^{-4} + 1}} \right) \quad (5c)$$

Notice that all the components of the energy momentum tensor are finite in the spacetime of the electric nonlinear monopole.

a) The weak energy condition (WEC):

$$T_{\mu\nu}\xi^{\mu}\xi^{\nu} \geq 0 \text{ } (\xi^{\mu} : \text{any timelike vector}) \Rightarrow \rho \geq 0, \quad \rho + p_k \geq 0 \quad , \quad (k = 1, 2, 3)$$

guarantees that the energy density as measured by any local observer is non-negative. For our case having account in expressions (4) and (5) :

$$\rho = T_{00} = \frac{b^2}{4\pi} \left(\sqrt{(\bar{Y})^{-4} + 1} - 1 \right) \geq 0$$

$$\rho + p_{rad} = 0$$

$$\rho + p_{\perp} = \frac{b^2}{4\pi} \left(\sqrt{(\bar{Y})^{-4} + 1} - \frac{1}{\sqrt{(\bar{Y})^{-4} + 1}} \right) \geq 0$$

Then, the WEC is satisfied in all the manifold.

b) The dominant energy condition (DEC):

includes WEC and requires each each principal pressure never exceeds the energy density which guarantees that the speed of sound cannot exceed the light velocity c

$$T^{00} \geq |T^{ik}|, \quad (i, k = 1, 2, 3) \Rightarrow \rho \geq 0, \quad \rho + p_k \geq 0$$

were probed before in a) and, from expressions (4) and (5)

$$\rho - p_{rad} = 2\rho \geq 0$$

$$\rho - p_{\perp} = \frac{b^2}{4\pi} \left(-2 + \sqrt{(\bar{Y})^{-4} + 1} + \frac{1}{\sqrt{(\bar{Y})^{-4} + 1}} \right) = \frac{b^2}{4\pi} \left(-2 + \frac{(\bar{Y})^{-4} + 2}{\sqrt{(\bar{Y})^{-4} + 1}} \right) \geq 0$$

the WEC is satisfied.

c) The strong energy condition (SEC):

$$\text{requires : } \rho + \sum p_k = \frac{b^2}{2\pi} \left(1 - \frac{1}{\sqrt{(\bar{Y})^{-4} + 1}} \right) \geq 0$$

and defines the sign of the acceleration due to gravity and is fulfilled in our case.

Notice because all the energy conditions are satisfied, not exotic matter-energy need to be introduced in order to explain any anomalous behaviour of the fields. Also in some references was claimed about that the metric coefficient take a de Sitter behaviour when $r \rightarrow 0$. And this fact is obvious: all regular solution necessarily has this form $1 - Ar^2$ (A : some constant factor) in order to avoid divergences produced by terms as $\frac{1}{r^n}$ with $n > 1$ near the origin.

3 Horizons: physical considerations

The metric, as was shown in ¹, is absolutely regular² being in our case, a static spherically symmetric spacetime (SSS) with $g_{tt} \neq g_{rr}^{-1}$. This fact make that the number of horizons in g_{tt} can be none, one (extreme) or two depending on the specific value of the parameters of the solution, but g_{rr} present only one extreme horizon (it is $g_{rr} = 0$). The localization of the minimum of $-g_{tt}$ coincides with the localization of the point $g_{rr} = 0$.

The physical interpretation is: because the form of the spacetime depends on the distribution of mass/energy, that in the point of maximum value of the electric field g_{rr} takes the minimum value possible that correspond with a spherical surface the radius $r_{cr} = -\frac{\sqrt[3]{2}}{a}r_o$ (at this point we are over the surface/border of the electric spherical configuration). However, this fact don't disturb the computation of the electromagnetic mass where the integration of the T_0^0 component (the energy) was made, then the identification between the electromagnetic and gravitational mass remains unaltered. Then, the r coordinate is well behaved wherever be the behaviour of g_{tt} (one, two or none horizons) because remains $g_{rr} \geq 0$ in all the manifold. When g_{tt} present 2 horizons there exists one region where the manifold takes Euclidean signature in the coordinate system the field loses its electric character into this region). The important thing for the analysis of this Section is that $g_{tt} = e^{2\Lambda}$ given by (2) is finite and regular without divergences or incompleteness of any type in all the manifold and g_{rr} presents a regular horizon at the radius corresponding to minimum vale of \bar{Y}^2 .

Line element Now, we introduce in the general line element (1) a set of null modified coordinates suitable to describe the characteristic null surface at the extreme horizon r_{extr} ($Y_{extr} = Y(r_{extr})$) in g_{rr} (angular part remaining the same)

$$\frac{du + dv}{2} = dt \quad , \quad \frac{dv - du}{2} = 2(f'(r) + 1)dr \quad (6)$$

$$f'(r) = \frac{[(a|r|)^n + (mn - 1)r_0][1 - (a|r|)^{-n}r_0]^m}{(a|r|)^n - r_0} - 1$$

²However, as was shown in ref.[1], the global properties of the spacetime are easily seen writting the line element (1) as a function of the $Y(r) : ds^2 = -e^{2\Lambda}dt^2 + e^{-2\Lambda}dY^2 + Y^2(d\theta^2 + \sin^2\theta d\varphi^2)$

$$\begin{aligned}
ds^2 &= -\frac{e^{2\Lambda}}{4} (du + dv)^2 + \frac{e^{-2\Lambda}}{4} (-du + dv)^2 \\
&= -\text{Sinh}(2\Lambda) \frac{(du^2 + dv^2)}{2} - \text{Cosh}(2\Lambda) dudv
\end{aligned}$$

In the coordinates (6) and with the values of the parameters fixed to $a = -0,9$ and $m = 1$ and $n = 3$, the line element is manifestly regular everywhere, presenting a good behaviour also for $r \rightarrow 0$ and $r \rightarrow \infty$ where the spacetime is flat ($\Lambda \rightarrow 0$). Also similar happens in the Schwarzschild line element when Kruskal type coordinates are introduced in $r = 2m$, but in our case this is not a maximal analytical extension because this SSS is itself complete and need not be extended..

Geodesic completeness From the geodesic point of view and for a static spherically symmetric spacetime (SSS), the study of the radial case is sufficient to analyze its completeness. Then, explicit computation of the radial geodesics lead to^{9,13,15}:

$$\frac{dt}{ds} = e^{-2\Lambda} C \quad (7)$$

$$\begin{aligned}
\frac{dr}{ds} &= \pm (\delta e^{-2\Phi} + e^{-2(\Lambda+\Phi)} C^2)^{1/2} \\
\frac{dr}{dt} &= \pm \frac{e^{2\Lambda}}{C} (\delta e^{-2\Phi} + e^{-2(\Lambda+\Phi)} C^2)^{1/2}
\end{aligned}$$

where as in ref.⁹ the constant C characterizes the test particle under consideration, and $C > 1$, $= 1$, < 1 correspond respectively to finite, zero and imaginary velocity of the particle at infinity (an infinite value of C represents a light ray $ds = 0$); and the constant $\delta = 1, 0$ for timelike or null geodesics respectively. From above equations is easily seen, that all non-spacelike geodesics can be extended to arbitrary values of the affine parameter:

i) Radial null geodesics:

from equations (7) with $\delta = 0$, we see that: $0 \leq \left| \frac{dt}{ds} \right| = \left| \frac{dr}{ds} e^{-\Lambda} \right| \leq |C|$, from where it follows that these geodesics are complete.

ii) Radial timelike geodesics:

from equations (7) with $\delta = 1$, we see that: $0 \leq \left| \frac{dt}{ds} \right| = \left| e^{2\Lambda} + \left(\frac{dr}{ds} \right)^2 e^{4\Lambda} \right|^{1/2} \leq \left| e^{2\Lambda} + C^2 \right|^{1/2}$, from where it follows that these geodesics are complete.

Curvature As was pointed out in ref.¹ and for the set of parameters fixed to the values: $a = -0,9$ and $m = 1$ and $n = 3$, the explicit computation of the curvature scalar indicate us that the spacetime is singularity free (geometrodynamics units ¹):

$$R_a^a = R = 2(G_{00} - G_{22}) = \left(-2 + \frac{\bar{Y}^{-4} + 2}{\sqrt{\bar{Y}^{-4} + 1}} \right)$$

that clearly shows that $R \rightarrow 0$ when $r \rightarrow 0, \infty$ and has not divergences in all the manifold³.

Extreme horizon localization In order to determine the horizon in the extreme case for g_{tt} (because we have been seen in our previous paper that we can have none, one or two horizons for g_{tt}), the b value must be fixed at the last stage of the computation. As we pointed out before, $g_{rr} = 0$ at some critical radius where the electric field take its maximum value and g_{tt} presents the maximum curvature. In the cases where there exist an extreme horizon of this type, as in the case presented here, the position of it can be find as follows:

- i) select a set of parameters a, m, n, r_0 , e.g. $a, 1, 3, r_0 \rightarrow Y^2 \equiv \left[1 - \left(\frac{r_0}{a|r|} \right)^3 \right]^2 r^2$
- ii) compute where is the extreme of the electric field, the metric or where $g_{rr} = 0$, e.g: $\frac{dF_{10}}{dr} = 0 \rightarrow -\frac{1}{2} = \left(\frac{r_0}{a|r|} \right)^3$
- iii) from the previous point, determine the critical values r_{cr} (or $Y_{cr}(r)$) $\rightarrow r_{cr} = -\frac{\sqrt[3]{2}}{a} r_0 \left(Y_{cr} = -3\frac{\sqrt[3]{2}}{2} \frac{r_0}{a} \right)$. For $a = -0.9$ and $r_0 = 1$ (our case): $r_{cr} = 1.39991$ ($Y_{cr} = 2.09987$)
- iv) with the critical values r_{cr} (or $Y_{cr}(r)$) in the expression for $g_{tt} = 0$ we determine the corresponding b value.

³Moreover, since that the tensor R_{bcd}^a for the spherically symmetric static metric under consideration here is pairwise diagonal, the Kretschmann scalar \mathcal{K} is a sum of squares:

$$\mathcal{K} = R^{ab}_{cd} R^{cd}_{ab} = 4(R^0_{110})^2 + 8(R^2_{112})^2 + 8(R^0_{220})^2 + 4(R^3_{223})^2$$

where the components of the Riemann tensor were explicit computed in I, that, with the corresponding values for the metric coefficients of the solution obtained in our previous work, indicate us that the metric presents no problems also.

Then for the determined b value the extreme horizon in g_{tt} is in $r = r_{cr}$ (Fig.1).

In resume, we saw that the metric coefficients are not reciprocals (different) because the line element is the more general spherically symmetric adecuated to the symmetries of the Born-Infeld theory. This fact is based in the observation that the Born-Infeld energy-momentum tensor takes the same form in the tetrad defined for expressions in I that an anisotropic fluid then, it defines that the correct ansatz to solve the problem¹⁵.

The spacetime is geodesically complete and singularity free: all non-spacelike geodesics can be extended to arbitrary values of the affine parameter and the curvature (and Kretschmann) scalar presents no divergencies for all values of r (see ref.⁹).

3.1 The absolute b field, the electron mass and the cosmological constant

In this Section we will to consider some physical aspects of some quantities naturally introduced in the BI theory: the absolute field b and the radius r_0 that was related in the earliest references^{2,3,4} with the electron radius.

From the mass formula (2.2) and having account that $b^2 r_o^4 = Q^2$ we have an expression for r_0 as a function of the mass and charge of the electron:

$$r_0 \cong 3.4831.10^{-13}cm$$

that is a more accurate value for the electron radius than the given in reference², or for logarithmic type Lagrangians as in¹⁹. Now, the value of the absolute field b is easily determined

$$b \cong 3.96718.10^{15}esu/cm^2$$

However, the enormous magnitude of this absolute b field justifies the application of the Maxwell's equations in their classical form in all cases, except those were the inner structure of the electron is concerned (fields of the order b , distance or wavelength of order r_0).

Is interesting to note that if we take the value of $b_{cr} = 3,56647.10^{24}esu/cm^2$ (the value of b where g_{tt} have the extreme horizon) and $Q \simeq e/3$ (as suggested time ago for Rosen as for the quarks case), the obtained value for r_0 is $6,7.10^{-18}cm$ given a mass of the order of $9,5.10^{-23}g \sim 5,33.10^4 GeV$.

Finishing this discussion about orders of magnitude and physical considerations; in the reference¹⁰ a new non-abelian generalization of the BI theory was proposed, from the first principles, with many interesting properties as to be absolutely independent of the symmetry gauge groups, then of the trace prescriptions¹¹ and presents a full generalized non-linear duality¹². A new wormhole solution was obtained solving the Einstein-NABI equations leading to the following ordinary differential equation for the scale factor a

$$3 \left[\left(\frac{\dot{a}}{a} \right)^2 - \frac{1}{a^2} \right] = 2G(b^2 - 4\pi\Lambda) - 2Gb^2 \left[1 + 6 \left(\frac{r_0}{a} \right)^4 \left(1 + \left(\frac{r_0}{a} \right)^4 \right) \right]^{1/2}$$

As was shown in¹¹ the integrability condition for this equation is (geometrodynamic units¹⁰)

$$(b^2 = 4\pi\Lambda)$$

this fact constrain the value of for the non-abelian case to $b \sim 1.23506 \cdot 10^{58} esu/cm^2$ and $r_0 \sim l_{Planck}$. It is obvious that we can consider, if the value of b is of the above order, an effective cosmological constant $\Lambda_{eff} \sim (b^2 - 4\pi\Lambda) \sim 0$ where the big value of Λ_{Planck} is screened by an absolute field associated with a generalized NABI theory. This fact can be of great importance in models based in d-brane/superstring theories where the b-field is naturally related with the d-brane (string) tension¹¹.

4 Regularity, inconsistencies and no-go theorems

As was clearly explained in reference¹, and we have been mention in the Introduction, the main point in order to obtain SSS in the (nonlinear) EBI theory are the following

i) Most general spherically symmetric line element as starting point, compatible with the symmetry of the an anisotropic fluid (the Born-Infeld energy-momentum tensor has this form), that lead a general solution as expression (1)

ii) strict physical requirements of regularity for the electric field:

The electric field cannot be greater that the absolute field b

$$F_{01}|_{r=r_o} < b$$

The electric field need to be zero at the origin⁴ $r \rightarrow 0$

$$F_{01}|_{r=0} = 0$$

and for $r \rightarrow \infty$ Coulombian behaviour

$$F_{01}|_{r \rightarrow \infty} = 0 \quad \text{asymptotically Coulomb}$$

The Lagrangian of the regular electric monopole of¹ is

$$L = \frac{b^2}{4\pi} \left(1 - \sqrt{\left[1 - \left(\frac{F_{01}}{b} \right)^2 \right]} \right)$$

where

$$\left(\frac{F_{01}}{b} \right)^2 = \frac{1}{Y^4 + 1}$$

never diverges having account the behaviour of $Y^2(r)$ for the selected values for the parameters [see (4)]

$$\begin{aligned} &\text{for } a = -0,9, m = 1 \text{ and } n = 3 \\ &\text{when } r \rightarrow 0, Y^2(r) \rightarrow \infty, \quad \text{and when } r \rightarrow \infty, Y^2(r) \rightarrow r^2 \end{aligned}$$

then

$$\text{when } r \rightarrow 0, \left(\frac{F_{01}}{b} \right)^2 \rightarrow 0, \text{ and when } r \rightarrow \infty, \left(\frac{F_{01}}{b} \right)^2 \rightarrow 0$$

and because the maximum value of the electric field never takes the absolute field b value e.g: $F_{01}|_{r=r_0} < b \Rightarrow L \rightarrow 0$ at $r \rightarrow 0$ that is the Maxwellian behaviour. Also:

$$\text{when } r \rightarrow 0, \frac{dL}{dS} = \sqrt{\left[1 - \left(\frac{F_{01}}{b} \right)^2 \right]} \rightarrow 1$$

where: $S \equiv -\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}$. Is easily seen that L as a function of F has not branches. Because the energy-momentum tensor is absolutely regular, as we show in expressions (5), and goes to zero when $r \rightarrow 0$, the density T_{00} goes

⁴r corresponding to a coordinate basis

obviously to zero (Fig.2) according to physical requirements of regularity. Schematically

$$F \rightarrow 0 \Rightarrow \rho \rightarrow 0 \Leftrightarrow g \rightarrow 1(\text{flat space, vacuum})$$

Notice that in particular SSS solutions from nonlinear electrodynamics⁷ the points i) and ii) were not having account and, in order to avoid divergences of $\frac{dL}{dS}$, a maximal density ρ at the centre of the spherical configuration was required.

To finish this Section, we show that the field

$$\mathbb{F}_{01} = \frac{F_{01}}{\sqrt{1 - (\overline{F}_{01})^2}}$$

that is related with the D-field of the an electrodynamics in a continuum medium²⁰ has similar regular well behaviour that the electric field as is easily seen from above equation (Figure 3). As far as we know, all the other solutions coming from NED formulations^{6,7,8} present divergencies of this D-field (\mathbb{F}_{01}) when $r \rightarrow 0$.

5 Dyon: statement of the problem

Is clearly important in order to finish our research, to consider the more general case that is the static spherically symmetric solution (SSS) in EBI theory with electric and magnetic field: the dyon. To do this, we propose the following line element for the static Born-Infeld monopole as in¹

$$ds^2 = -e^{2\Lambda} dt^2 + e^{2\Phi} dr^2 + e^{2F(r)} d\theta^2 + e^{2G(r)} \sin^2 \theta d\varphi^2 \quad (8)$$

where the components of the metric tensor are

$$\begin{aligned} g_{tt} &= -e^{2\Lambda} & g^{tt} &= -e^{-2\Lambda} \\ g_{rr} &= e^{2\Phi} & g^{rr} &= e^{-2\Phi} \\ g_{\theta\theta} &= e^{2F} & g^{\theta\theta} &= e^{-2F} \\ g_{\varphi\varphi} &= \sin^2 \theta e^{2G} & g^{\varphi\varphi} &= \frac{e^{-2G}}{\sin^2 \theta} \end{aligned} \quad (9)$$

For the obtention of the Einstein-Born-Infeld equations system we use the Cartan's structure equations method¹, that is most powerful and direct where

we work with differential forms and in a orthonormal frame (tetrad). The line element (7) in the 1-forms basis takes the following form

$$ds^2 = -(\omega^0)^2 + (\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2 \quad (10)$$

were the forms are

$$\begin{aligned} \omega^0 &= e^\Lambda dt & \Rightarrow & dt = e^{-\Lambda} \omega^0 \\ \omega^1 &= e^\Phi dr & \Rightarrow & dr = e^{-\Phi} \omega^1 \\ \omega^2 &= e^{F(r)} d\theta & \Rightarrow & d\theta = e^{-F(r)} \omega^2 \\ \omega^3 &= e^{G(r)} \sin \theta d\varphi & \Rightarrow & d\varphi = e^{-G(r)} (\sin \theta)^{-1} \omega^3 \end{aligned} \quad (11)$$

Now, following the standard procedure of the structure equations (Appendix in ref.¹) for to obtain easily the components of the Riemann tensor, we can construct the Einstein equations

$$G^1{}_2 = -e^{-(F+G)} \frac{\cos \theta}{\sin \theta} \partial_r (G - F) \quad (12)$$

$$G^0{}_0 = e^{-2\Phi} \Psi - e^{-2F} \quad (13)$$

$$\Psi \equiv [\partial_r \partial_r (F + G) - \partial_r \Phi \partial_r (F + G) + (\partial_r F)^2 + (\partial_r G)^2 + \partial_r F \partial_r G]$$

$$G^1{}_1 = e^{-2\Phi} [\partial_r \Lambda \partial_r (F + G) + \partial_r F \partial_r G] - e^{-2F} \quad (14)$$

$$G^2{}_2 = e^{-2\Phi} [\partial_r \partial_r (\Lambda + G) - \partial_r \Phi \partial_r (\Lambda + G) + (\partial_r \Lambda)^2 + (\partial_r G)^2 + \partial_r \Lambda \partial_r G] \quad (15)$$

$$G^3{}_3 = e^{-2\Phi} [\partial_r \partial_r (F + \Lambda) - \partial_r \Phi \partial_r (F + \Lambda) + (\partial_r \Lambda)^2 + (\partial_r F)^2 + \partial_r F \partial_r \Lambda] \quad (16)$$

$$G^1{}_3 = G^2{}_3 = G^0{}_3 = G^0{}_2 = G^0{}_1 = 0 \quad (17)$$

In the tetrad defined by (11), the energy-momentum tensor of Born-Infeld takes a diagonal form, being its components the following

$$-T_{00} = T_{11} = \frac{b^2}{4\pi} \left(\frac{\mathbb{R} - 1}{\mathbb{R}} \right) \quad (18)$$

$$T_{22} = T_{33} = \frac{b^2}{4\pi} (1 - \mathbb{R}) \quad (19)$$

where for the dyon

$$\mathbb{R} \equiv \sqrt{\left[1 - \left(\frac{F_{01}}{b}\right)^2\right] \left[1 - \left(\frac{F_{32}}{b}\right)^2\right]} \quad (20)$$

of this manner, one can see from the Einstein equation (12) the characteristic property of the spherically symmetric space-times¹⁵

$$G^1{}_2 = -e^{-(F+G)} \frac{\cos \theta}{\sin \theta} \partial_r (G - F) = 0 \quad \Rightarrow \quad G = F \quad (21)$$

Notice for that the interval be a spherically symmetric one, the functions $F(r)$ and $G(r)$ must be equal. As we saw in the precedent paragraph the components of the energy-momentum tensor of BI assures this condition in a natural form. Also it is interesting to see from eqs. (18) and (19) that the energy-momentum tensor of Born-Infeld has the same form as the energy-momentum tensor of an anisotropic fluid¹⁵.

6 Dyons: field equations vs. energy-momentum conservation

To obtain explicitly the electromagnetic fields in the tetrad (11), the following equations need to be solved

$$dF = 0 \quad ; \quad d\tilde{F} = 0$$

the Bianchi identity and the equations of motion for the electromagnetic field, that in the same language that in¹, are

$$\nabla_a \mathbb{F}^{ab} = \nabla_a \left[\frac{F^{ab}}{\mathbb{R}} + \frac{P}{b^2 \mathbb{R}} \tilde{F}^{ab} \right] = 0 \quad (\text{field equations}) \quad (22)$$

$$\nabla_a \tilde{F}^{ab} = 0 \quad (\text{Bianchi's identity}) \quad (23)$$

where

$$P \equiv -\frac{1}{4} F_{\alpha\beta} \tilde{F}^{\alpha\beta} \quad (24)$$

$$S \equiv -\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} \quad (25)$$

$$\mathbb{R} \equiv \sqrt{1 - \frac{2S}{b^2} - \left(\frac{P}{b^2}\right)^2} = \sqrt{\left[1 - \left(\frac{F_{01}}{b}\right)^2\right] \left[1 + \left(\frac{F_{32}}{b}\right)^2\right]} \quad (26)$$

The above equations can be solved explicitly giving the follow result

$$dF = 0 \Rightarrow F_{01}e^{\Lambda+\Phi} = A(r) \quad \text{and} \quad F_{23}e^{2G} = B$$

and

$$\widetilde{d\mathbb{F}} = 0 \Rightarrow \mathbb{F}_{23} = F_{23} \sqrt{\frac{1 - \left(\frac{F_{01}}{b}\right)^2}{1 + \left(\frac{F_{32}}{b}\right)^2}} = C(r) e^{-(\Lambda+\Phi)} \quad \text{and} \quad (26.1)$$

$$F_{01} \sqrt{\frac{1 + \left(\frac{F_{32}}{b}\right)^2}{1 - \left(\frac{F_{01}}{b}\right)^2}} = D e^{-2G} \quad (26.2)$$

From the above to equations is easily seen that

$$u \equiv \sqrt{\frac{1 + \left(\frac{F_{32}}{b}\right)^2}{1 - \left(\frac{F_{01}}{b}\right)^2}} = \frac{D}{A(r)} = \frac{B}{C(r)} \quad (27)$$

then, without lost generality, we can make $B = D$ and $A(r) = C(r)$.

Now is very early to say something about the constants, then, the real form of the electromagnetic field. In order to corroborate and restrict the kind of solutions we must go to the equations of the energy-momentum conservation in the tetrad (11)

$$\nabla_a T^{ab} = 0 \quad (28)$$

$$\nabla_1 T^{11} = 0 \rightarrow \partial_r T^{11} + 2\partial_r G (T^{11} - T^{22}) = 0$$

$$\nabla_1 T^{11} \equiv 0, \quad \nabla_3 T^{33} = \nabla_2 T^{22} = 0$$

where the non vanishing connection coefficients from the tetrad defined by (11) are easily obtained from $\Gamma_{cd}^b = \langle \omega^b, \nabla_d E_c \rangle$ as usual.

Because the components of the energy-momentum tensor in the tetrad (11) can be explicitly written as functions of the u invariant (27)

$$-T_{00} = T_{11} = \frac{b^2}{4\pi} (1 - u), \quad T_{22} = T_{33} = \frac{b^2}{4\pi} (1 - u^{-1}) \quad (29)$$

the final form of u is immediately determined

$$u = \sqrt{\left(\frac{r_0}{e^G}\right)^4 + 1} \quad (30)$$

that is the expected result with r_0 the integration constant with length units that was related with the radius of the electron in ref.². Notice that from the point of view of the energy-momentum conservation only the combination of the electromagnetic invariants u is determined. Also, as pointed out in², this is a direct consequence of the Von Laue's theorem in an *unitarian electromagnetic theory* as is the Born-Infeld case.

Now, from eqs.(27), (26.1) and (26.2) and considering $F_{23} = De^{-2G}$ obtained from the field equations(notice an scale arbitrariness in the radius $D/A(r)$) we arrive to

$$F_{01} = b\sqrt{\frac{1 - (D^2/b^2r_0^4)}{1 + (e^{4G}/r_0^4)}} \quad (31)$$

that is also an expected result considering that one recover the expression for the electric monopole putting $D = 0$. In the same manner that for the pure electric case we can suppose $D = Q_m$ and $Q_T = br_0^2$ (because the units allows us). Then

$$F_{01} = b\sqrt{\frac{Q_T^2 - Q_M^2}{Q_T^2 + (b^2e^{4G})}} \quad \text{and} \quad F_{23} = e^{-2G}Q_m \quad (32)$$

Is important to observe that the concept of charge is only an asymptotic idea. The last expressions of the electromagnetic fields as functions of charges (that doesn't exist! they are there due by the nonlinearity of the theory) is only a "far away" hint in order to interpret this unitarian theory as in the maxwellian (linear and dualistic) case. In resume:

i) for the dyonic solution the metric will have the same behaviour and properties that the electric case previously studied by the authors in reference¹: is obvious that the Einstein equations remain the same although the presence of the electric and magnetic fields.

ii) because the concept of charge is connected with the nonlinear behaviour of the electromagnetic fields, the same electric and magnetic components are interrelated as is easy to see in $F_{01} = b\sqrt{\frac{1 - (D^2/b^2r_0^4)}{1 + (e^{4G}/r_0^4)}}$ that depends on the magnetic part due the constant $D = Q_m$.

iii) from ii) is obvious that the one must necessarily forget the idea of charge as Q_m, Q_T etc being the correct expression for the fields

$$F_{01} = b \sqrt{\frac{1 - (D^2/b^2 r_0^4)}{1 + (e^{4G}/r_0^4)}} \quad \text{and} \quad F_{23} = e^{-2G} D \quad (33)$$

7 Dyon solution: the electromagnetic field

As was pointed out previously, the energy momentum tensor only take account on the electromagnetic fields content, e.g. u (27), through r_0 then, the Einstein equations for the electric, magnetic and dyonic cases are formally the same expressions. However, the interpretation of the charges, the fields and maximum values will be different in each case.

Regarding the electric and magnetic cases (e.g. Figure 1 of ¹ and Figure 4 in this paper) there are regular with coulombian behaviour at $r \rightarrow \infty$ and going to $\rightarrow 0$ when $r \rightarrow 0$ presenting the maximum value at the spherical surface with $r_B = 2^{1/3} \frac{r_0}{|a|}$.

The other important point is that the \mathbb{F}_{ab} field (called in some references as D-field^{6,7,8}) is absolutely regular in $r \rightarrow 0$ (Figure 3) presenting an analog behaviour as F_{ab} , remaining under the maximum value b in all the spacetime. This point is important because, as far as we known, all the solutions of non-linear electrodynamics present a divergence of \mathbb{F}_{ab} and also in the magnetic case^{5,6,7}, and many speculations were given about the impossibility of simultaneous regularity of \mathbb{F} and F . But the problem is solved having account the conditions pointed out previously in Section 4.

The dyonic case has a mixing between the pure electric and magnetic situation. The total intensity attributed to the spherical configuration is distributed between the electric and magnetic fields as dictated by equations (33). The electric field depends on the value of the magnetic field (this is the reason because the meaning of the word "charge" in this case is obscure). Also in this case the electric, magnetic and \mathbb{F} field are absolute regular at the origin and remaining all of them bounded by below the b value.

8 Discussion

Contrarily to the claimed in ref.⁸ we shown a SSS solution of the Born-Infeld theory that have regular behaviour without divergencies of the \mathbb{F} field at

$r=0$ and cusps or branches in the Lagrangian. The main reason because the circumvention of these claims^{6,7,8} is possible, is that we start from a more general SSS line element with $g_{tt} \neq g_{rr}^{-1}$ and strict regularity conditions over the electromagnetic field

$$\begin{aligned} F|_{r=r_o} &< b \\ F|_{r=0} &= 0 \end{aligned}$$

this fact permit to introduce the correct regularity conditions over the fields of the theory given the good features described in our previous work¹ and the analyzed here:

i) the field $\mathbb{F} \equiv \frac{\partial L}{\partial F}$ (D-field in) has regular behavior in $r \rightarrow 0$ (Figure 3)
ii) the metric is continuous at the border of the spherical configuration r_B : there are not "inner and outer" matching conditions in this limit for the solution, only one continuous function (i.e.Fig.1).

iii) from the point of view of the geodesics: the spacetime is geodesically complete and not continuation is required.

iv) the extreme horizon $g_{rr} = 0$ is absolutely regular, being a null surface precisely the sphere $r_B = 2^{1/3} \frac{r_o}{|a|}$, as is easily seen when in the line element modified null coordinates are specially introduced.

v) The physical interpretation is, because the form of the spacetime depends on the distribution of mass/energy, in the point of maximum value of the electric field g_{rr} takes the minimum value physically possible that correspond with an spherical surface the radius $r_{cr} = -\frac{\sqrt[3]{2}}{a} r_o$ (at this point we are over the surface/border of the spherical configuration of the electric field)

vi) the solution with the integration constant $M \neq 0$ is still regular due the behaviour of the $Y(r)$ with $-1 < a < 0$ (e.g. : $a = -0,9$; $m = 1$ and $n = 3$). However with $M \neq 0$,

- we cannot identify the electromagnetic with the gravitational mass;
- the solution is not invariant under the change $r \leftrightarrow -r$;
- when $r_o \rightarrow 0$ (Maxwell/Reissner-Nordström limit) the regularity is obviously broken.

vii) the dyonic case is analog to the electric and magnetic cases, but the electric field depends on the magnetic field (as dictated by equations (33)) being, in some sense, modulated by it. The total intensity attributed to the dyonic spherical configuration is distributed between the electric and magnetic fields. The metric have the same regular shape in the three situations: the energy-momentum tensor (then the Einstein equations (12-17)) don't see the specific structure of the magnetic and electric components of the F_{ab} : the

specific structure of the electromagnetic fields are determined by the dynamical field equations plus the conservation of the energy momentum tensor, as was shown in Section 6.

viii) the BI theory is *unitarian*: the concept of charge is asymptotically attributed due the non-linear behaviour of the electromagnetic field: the electromagnetic field is the basic identity. Then, the proposal of a "sum rule" of Q_e^2 and Q_m^2 due to duality prescriptions coming from the Maxwell (linear) theory is only an approximation, and we prefer present the fields with the correct constants as were obtained. The full symmetries of the BI field equations that give a more accurate answer to this point of general duality and charges was investigated in ref.¹² by the authors, and will be analyzed elsewhere¹⁴.

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